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Scalar plane waves in general relativity

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Abstract. A number of exact solutions of Einstein's equations are obtained, which describe the collisions between one scalar plane wave and one scalar, neutrino, electromagnetic or gravitational plane wave.

1. Introduction

Nonlinearity in general relativity is most clearly shown by the interaction of two gravitational waves. Kahn and Penrose (1970) and Szekeres (1970) independently discovered the first exact solution representing the collision and found that the focusing effect of each wave on the other results in a singularity in space–time. Later, Bell and Szekeres (1974) found a solution for the gravitational interaction of two electromagnetic waves. Gravitational impulse waves are produced as a result of two electromagentic shock waves in collision. Griffiths (1976a, b) has obtained an exact solution for two neutrino waves and considered collisions between any two types of gravitational, electromagnetic and neutrino waves.

The problem treated in this paper consists of the gravitational collisions between two complex, massless, scalar plane waves. The motivation for this problem is as follows: it is believed that first-order phase transitions from an old false vacuum to a new real vacuum occurred in the very early universe. Bubbles of the real vacuum would have formed and expanded. The complex scalar field in any two bubbles of real vacuum must have the same magnitude but can have arbitrary phase. When two bubbles of different phase collide, two phase waves propagate out from the collision region at the speed of light (Hawking *et al* 1981). These phase waves can each be approximated by a complex scalar, massless plane wave. The physical effects of collisions between phase waves from different bubble collisions may be as important as the bubble collisions themselves. For completeness we also give a number of exact solutions of Einstein's equations which describe the collisions between one scalar plane wave and one neutrino, electromagnetic or gravitational plane wave. We use the method of Szekeres and Griffiths (Szekeres 1972, Griffiths 1976b). Following them, we derive the field equations in terms of the Newman–Penrose formalism (Newman and Penrose 1962) in § 2. In § 3 an exact solution representing two interacting pure scalar plane waves is derived. Section 4 deals with the collisions between one scalar wave and a gravitational, electromagnetic or neutrino wave.

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2. The field equations

We may, without loss of generality, consider the 'head-on' collision of two waves, since an appropriate Lorentz transformation can always be found to make the incident waves approach from exactly opposite spatial directions. We choose space-time coordinates (u, v, x, y) where x, y are space-like coordinates in the surface perpendicular to the direction of motion and u, v are null. The equations $u=\text{constant}$ and $v=\text{constant}$ define the propagation of the two waves. It is assumed that they collide at $u=0,\ v=0$.

In region I $(u<0, v<0)$ the space-time is flat,

$$ds^2 = 2\,du\,dv - (dx^2+dy^2). \tag{2.1}$$

In region II $(u>0, v<0)$ a plane wave propagates along $u=\text{constant}$. It is convenient to transform the metric into Rosen (1931) form

$$ds^2 = 2e^{-M}\,du\,dv - g_{xx}\,dx^2 - 2g_{xy}\,dx\,dy - g_{yy}\,dy^2 \tag{2.2}$$

and all field quantities and M, g_{ij} are functions of u.

Similarly in region III $(u<0, v>0)$ another plane wave propagates along $v=$ constant. All physical quantities can be expressed in the same way as in II, by exchanging u and v.

In region IV $(u>0, v>0)$, after the collision of the two wavefronts we assume that the metric can be expressed in the form (Szekeres 1974)

$$ds^2 = 2e^{-M}\,du\,dv - e^{-V}(e^{V}\cosh W\,dx^2 + e^{-V}\cosh W\,dy^2 - 2\sinh W\,dx\,dy) \tag{2.3}$$

where the field quantities and U, V, M, W, are all functions of (u, v). V, W represent the polarisation of gravitational disturbance. If $V=W=0$, then the wave plane has plane symmetry ISO(2).

Using Newman–Penrose formalism, the Einstein field equations can be expressed as

$$U_{uv} = U_uU_v - 2\Phi_{11} - 6\Lambda, \tag{2.4}$$

$$2U_{vv} = U_v^2 + W_v^2 + V_v^2\cosh^2 W - 2U_vM_v + 4\Phi_{00}, \tag{2.5}$$

$$2U_{uu} = U_u^2 + W_u^2 + V_u^2\cosh^2 W - 2U_uM_u + 4\Phi_{22}, \tag{2.6}$$

$$2M_{uv} = -U_uU_v + W_uW_v + V_uV_v\cosh^2 W + 8\Phi_{11}, \tag{2.7}$$

$$2W_{uv} = U_uW_v + U_vW_u + 2V_uV_v\sinh W\cosh W + 2i(\Phi_{02}-\Phi_{20}), \tag{2.8}$$

$$(2V_{uv} - U_uV_v - U_vV_u)\cosh W + 2(V_vW_u + V_uW_v)\sinh W = 2(\Phi_{02}+\Phi_{20}). \tag{2.9}$$

The components of the Weyl tensor, which tell us whether gravitational radiation is present, are

$$\Psi_0 \propto -\tfrac{1}{2}[V_{vv}\cosh W + 2V_vW_v\sinh W - V_v(U_v - M_v)\cosh W] + \tfrac{1}{2}i[W_{vv} - W_v(U_v - M_v) - V_v^2\cosh W\sinh W] \tag{2.10}$$

$$\Psi_4 \propto -\tfrac{1}{2}[V_{uu}\cosh W + 2V_uW_u\sinh W - V_u(U_u - M_u)\cosh W] - \tfrac{1}{2}i[W_{uu} - W_u(U_u - M_u) - V_u^2\cosh W\sinh W], \tag{2.11}$$

$$\begin{aligned}&\Psi_2 = e^{-M}[\tfrac{1}{2}M_{uv} - \tfrac{1}{2}i(V_uW_v - V_vW_u)\cosh W],\\ &\Psi_1 = \Psi_3 = 0.\end{aligned} \tag{2.12}$$

For all the cases considered in this paper we have $\Phi_{02}=\Phi_{20}$ and $\Phi_{11}+3\Lambda=0$, hence, for simplicity, we can assume $W=0$ which implies that the polarisations of two waves match up in such a way that the metric can be simultaneously diagonalised in regions I, II, III and thus in region IV also (from the hyperbolic property of equation (2.8)). From equation (2.4) we obtain

$$U=-\ln(f(u)+g(v)), \qquad f(0)=g(0)=\tfrac{1}{2}, \tag{2.13}$$

where $f(u)$ and $g(v)$ are arbitrary functions.

It should be noticed that equation (2.9) is just the integrability condition for equations (2.4)–(2.7). Thus we may concentrate on solving equation (2.9) and the Euler–Lagrange equations for scalar, neutrino or electromangetic fields respectively, obtaining M from (2.5) and (2.6) by simple integration.

3. Colliding scalar waves

Since the energy–momentum tensor of a complex, massless, scalar field is

$$T_{ab}=\tfrac{1}{2}(\varphi_a\bar{\varphi}_b+\varphi_b\bar{\varphi}_a)-\tfrac{1}{2}g_{ab}\varphi_c\bar{\varphi}^c, \tag{3.1}$$

we can set $\varphi=\varphi_0$ (constant) in region I and $\varphi=\varphi(u)(\varphi(v))$ in region II (III). The only non-vanishing Ricci tensor components are

$$\Phi_{00}=-\tfrac{1}{2}\varphi_1\bar{\varphi}_1, \qquad \Phi_{22}=-\tfrac{1}{2}\varphi_2\bar{\varphi}, \qquad \Phi_{11}=-\tfrac{1}{8}(\varphi_1\bar{\varphi}_2+\varphi_2\bar{\varphi}_1)=-\Lambda/3. \tag{3.2}$$

φ must satisfy the scalar wave equation

$$\varphi_{uv}-\varphi_u U_v-\varphi_v U_u=0. \tag{3.3}$$

Comparing equations (2.4), (2.9) with (3.5), it is interesting to notice that U, V evolve in exactly the same way as the scalar field potential. To have incident waves which are purely scalar field waves, we can set $V=0$ in regions I, II, III (so from the hyperbolic property of equation (2.9) $V=0$ in region IV), causing all the components of the Weyl tensor to vanish in regions I, II and III. By a change of variables, $f=f(u)$, $g=g(v)$ the scalar wave equation (3.3) can be transformed into an Euler–Darboux equation

$$2(f+g)\varphi_{fg}+\varphi_f+\varphi_g=0. \tag{3.4}$$

In principle, it is possible to find the solution to equation (3.4) in region IV, using the Riemann–Green function method (Courant and Hilbert 1961), with given boundary conditions at $u=0$, $v>0$ and $v=0$, $u>0$, i.e. with known incident waves. However, more simply, by trial and error one finds the following explicit solution

$$\varphi=k_1\tan^{-1}\left(\frac{f-\frac{1}{2}}{g+\frac{1}{2}}\right)^{1/2}+k_2\tan^{-1}\left(\frac{g-\frac{1}{2}}{f+\frac{1}{2}}\right)^{1/2}+\varphi_0 \qquad (k_1, k_2, \varphi_0 \text{ complex}). \tag{3.5}$$

In region II we must have $\varphi=\varphi_1(f)=k_1\tan^{-1}(f-\tfrac{1}{2})^{1/2}$, and $\varphi=\varphi_2(g)=k_2\tan^{-1}(g-\tfrac{1}{2})^{1/2}$ in region III. The complete solution is expressed most compactly by putting

$$p=(f-\tfrac{1}{2})^{1/2}, \qquad q=(g-\tfrac{1}{2})^{1/2} \qquad r=(\tfrac{1}{2}+f)^{1/2}, \qquad w=(\tfrac{1}{2}+g)^{1/2}, \qquad t=(f+g)^{1/2}, \tag{3.6}$$

where

$$f=\tfrac{1}{2}+(au)^{n_1}\theta(u), \qquad g=\tfrac{1}{2}+(bv)^{n_2}\theta(v) \qquad (a, b, n_1, n_2 \text{ real})$$

and $\theta(x)$ is the Heaviside step function. Then

$$\varphi=k_1 \tan^{-1}\frac{p}{w}+k_2 \tan^{-1}\frac{q}{r}+\varphi_0. \tag{3.7}$$

Integration of equations (2.5) and (2.6) results in

$$|k_i|^2=4(1-1/n_i)$$

and

$$M=[1-\tfrac{1}{2}(k_1-k_2)(k_1^*-k_2^*)]\ln t+\tfrac{1}{2}|k_2|^2\ln\gamma+\tfrac{1}{2}|k_1|^2\ln w$$
$$-\tfrac{1}{2}(k_1k_2^*+k_2k_1^*)\ln(pq+rw). \tag{3.8}$$

The space–time is Petrov type D in region IV and, by assumption, conformally flat in regions II and III.

4. Collisions between scalar wave and other fields

In this section collisions between a scalar wave and a gravitational, electromagnetic or neutrino wave are considered.

For the first case, let region II contain a plane scalar wave $\varphi=\varphi(u)$ and region III contain a gravitational wave $\Psi_0=\Psi_0(v)$. The field equations remain the same as equations (2.4)–(2.9) and (3.4). It is easy to see that the following solution satisfies the required boundary conditions:

$$\varphi=k_1\tan^{-1}\left(\frac{f-\frac{1}{2}}{g+\frac{1}{2}}\right)^{1/2}+\varphi_0, \qquad (k_1, \phi_0 \text{ complex}) \tag{4.1}$$

$$V=k_2\tanh^{-1}\left(\frac{\frac{1}{2}-g}{f+\frac{1}{2}}\right)^{1/2} \qquad (k_2 \text{ real}) \tag{4.2}$$

where

$$f=\tfrac{1}{2}+(au)^{n_1}\theta(u), \qquad g=\tfrac{1}{2}-(bv)^{n_2}\theta(v), \qquad (a, b, n_1, n_2 \text{ real}).$$

As before these expressions can be simplified by putting

$$p=(f-\tfrac{1}{2})^{1/2}, \qquad q'=(\tfrac{1}{2}-g)^{1/2}, \qquad r=(\tfrac{1}{2}+f)^{1/2},$$
$$w=(\tfrac{1}{2}+g)^{1/2}, \qquad t=(f+g)^{1/2}. \tag{4.3}$$

In order that the Lichnerowicz (1955) conditions (the metric and its normal derivatives are continuous at the boundary) are satisfied at $v=0$, we must have $n_2\geqslant 1$. Integration of equations (2.5) and (2.6) results in

$$|k_1|^2=4(1-1/h_1), \qquad k_2^2=8(1-1/n_2),$$
$$M=(1-\tfrac{1}{2}k_1k_1^*-\tfrac{1}{4}k_2^2)\ln t+\tfrac{1}{4}k_2^2\ln r+\tfrac{1}{2}(k_1)^2\ln w. \tag{4.4}$$

In regions I, II space–time is flat and conformally flat respectively. In region III, the only non-vanishing component of the Weyl tensor Ψ_0 represents (i) $n_2=4$

($k_2=-\sqrt{6}$): gravitational shock wave $\Psi_0 \propto \theta(v)(\frac{1}{2}+g)^{-4}$, (ii) $n_2=2$ ($k_2=-2$): gravitational impulse wave $\Psi_0 \propto \delta(v)$.

For the second case region III contains an incident electromagnetic wave which contributes to the Ricci tensor by

$$\Phi_{AB}=\Phi_A\Phi_B \tag{4.5}$$

where the quantities Φ_i define the electromangetic field in Newman–Penrose formalism.

Maxwell's equation takes the form (Bell and Szekeres 1974)

$$\Phi_{1,v}=U_v\Phi_1, \qquad \Phi_{1,u}=U_u\Phi_1,$$
$$\Phi_{2,v}=-\tfrac{1}{2}V_u\Phi_0+\tfrac{1}{2}U_v\Phi_2, \qquad \Phi_{0,u}=-\tfrac{1}{2}V_v\Phi_2+\tfrac{1}{2}U_u\Phi_0. \tag{4.6}$$

For simplicity, we assume $\Phi_1=0$, $V=0$ and $\Phi_2=0$. These are necessary conditions to keep the space–time conformally flat before collision and electromagnetic-free in region II. Equation (4.6) implies

$$\Phi_0=F(g)/(f+g)^{1/2}, \qquad F(\tfrac{1}{2})=0, \tag{4.7}$$

where $F(g)$ is an arbitrary function. It is readily verified that the following constitute a solution of Einstein's equation and the scalar wave equation:

$$\varphi=k_1\tan^{-1}\left(\frac{f-\frac{1}{2}}{\frac{1}{2}+g}\right)^{1/2}+\varphi_0, \qquad (k_1, \phi_0 \text{ complex})$$

$$M=(1-\tfrac{1}{2}k_1k_1^*)\ln t+2(1-1/n_2)\ln q'-2\int F^2(g)\,\mathrm{d}g+\tfrac{1}{2}k_1k_1^*\ln w,$$

$$|k_1|^2=4(1-1/n_1),$$

where f, g, q', w, t, k_1, n_2 are defined as before.

For the last case, region III contains an incident neutrino plane wave. The neutrino field can be expressed in spinor form

$$\phi_A=\phi o_A+\psi\iota_A.$$

The neutrino Weyl equation takes the form (Griffiths 1976b)

$$\phi_{,v}=\tfrac{1}{2}U_v\phi, \qquad \psi_{,u}=\tfrac{1}{2}U_u\psi. \tag{4.8}$$

The components of the Ricci tensor due to the neutrino field become (Griffiths 1976b)

$$\Phi_{00}=\mathrm{i}(\psi\bar{\psi}_{,v}-\bar{\psi}\psi_{,v})$$
$$\Phi_{01}=\tfrac{1}{2}\mathrm{i}[(\bar{\phi}\psi)_{,v}-(\tfrac{3}{2}U_v-\tfrac{1}{2}M_v)\bar{\phi}\psi-\tfrac{1}{2}V_v\phi\bar{\psi}],$$
$$\Phi_{02}=-\mathrm{i}(-\tfrac{1}{2}V_v\phi\bar{\phi}+V_u\psi\bar{\psi}), \qquad \Phi_{11}=0,$$
$$\Phi_{12}=\tfrac{1}{2}\mathrm{i}[-(\bar{\phi}\psi)_{,u}-(-\tfrac{3}{2}U_u+\tfrac{1}{2}M_u)\bar{\phi}\psi+\tfrac{1}{2}V_u\phi\bar{\psi}]$$
$$\Phi_{22}=\mathrm{i}(\phi\bar{\phi}_{,u}-\bar{\phi}\phi_{,u}).$$

In order that the metric can take the Rosen form, the neutrino field must satisfy (Griffiths 1976a)

$$\phi\psi=0$$

which implies that the neutrino field is not reflected by the scalar field. We obtain the following explicit solution of the scalar wave, neutrino wave and Einstein equations

$$V=0,$$

$$\psi=\frac{F(g)}{(f+g)^{1/2}}, \qquad (F(\tfrac{1}{2})=0) \qquad \phi=0,$$

$$\varphi=k_1\tan^{-1}\left(\frac{f-\frac{1}{2}}{\frac{1}{2}+g}\right)^{1/2}+\varphi_0, \qquad k_1=4(1-1/n_1) \qquad (k_1,\, g_0 \text{ complex})$$

$$M=(1-\tfrac{1}{2}k_1k_1^*)\ln t+\tfrac{1}{2}k_1^*k_1\ln w+2(1-1/n_2)\ln q'-2\mathrm{i}\int F^2(g)\left(\frac{\bar{F}(g)}{F(g)}\right)_{,g}\mathrm{d}g,$$

where f, g, q', w, t, k_1, n_2, n_1 are defined as before and $F(g)$ is an arbitrary function.

We believe all these solutions to be new.

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